CO6 “Introduction to Computational Neuroscience”

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course website: http://iec-lnc.ens.fr/group-for-neural-theory/teaching-260/article/co6-course

Exercise Sheet 6 — 2 May 2017
Please submit your solution in the next class (9 May 2017)

(1) **Covariance and Correlation:** Assume that we have recorded two neurons in the two-alternative-forced choice task discussed in class. We denote the firing rate of neuron 1 in trial i as $r_{1,i}$ and the firing rate of neuron 2 as $r_{2,i}$. We furthermore denote the averaged firing rate of neuron 1 as $\bar{r}_1$ and of neuron 2 as $\bar{r}_2$. Let us “center” the data by defining two data vectors

$$\mathbf{x} = (r_{1,1} - \bar{r}_1\,,\, r_{1,2} - \bar{r}_1\,,\, r_{1,3} - \bar{r}_1\,,\, \ldots,\, r_{1,N} - \bar{r}_1)$$
$$\mathbf{y} = (r_{2,1} - \bar{r}_2\,,\, r_{2,2} - \bar{r}_2\,,\, r_{2,3} - \bar{r}_2\,,\, \ldots,\, r_{2,N} - \bar{r}_2)$$

(a) Show that the variance of the firing rates of the first neuron is $\mathrm{Var}(r_1) = \frac{1}{N}||\mathbf{x}||^2$.

(b) Compute the cosine of the angle between $\mathbf{x}$ and $\mathbf{y}$. What do you get?

(c) What are the maximum and minimum values that the correlation coefficient between r_1 and r_2 can take? Why?

(d) What do you think the term “centered” refers to?

(2) **Bayes theorem:** The theorem of Bayes summarizes all the knowledge we have about about the stimulus by observing the responses of a set of neurons, *independent* of a specific decoding rule. To get a better intuition about this theorem, we will look at the motion discrimination task again, and compute the probability that the stimulus moved to the left (+) or right (-). For a stimulus $s = \{+, -\}$, and a firing rate response r of the neuron, the theorem of Bayes reads

$$p(s|r) = \frac{p(r|s)p(s)}{p(r)} \quad .$$

Here, $p(r|s)$ is the probability that the firing rate is r if the stimulus was s. The respective distribution can be measured and we assume that it follows a Gaussian probability density with mean μ_s and standard deviation σ,

$$p(r|s) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(r - \mu_s)^2}{2\sigma^2}\Big)$$

The relative frequency with which the stimuli (leftward or rightward motion, $+$ or $-$) appear is denoted by $p(s)$, often called the *prior* probability or, for short, the *prior.* The distribution $p(r)$ denotes the probability of observing a response r, independent of any knowledge about the stimulus.

(a) How can you calculate $p(r)$?

(b) The distribution $p(s|r)$ is often called the *posterior* probability or, for short, the *posterior.* Calculate the posterior for the stimulus $s = +$. Sketch the posterior $p(s = +|r)$, or, for short, $p(+|r)$ as a function of r, assuming a prior $p(+) = p(-) = 1/2$. Draw the posterior $p(-|r)$ into the same plot.

(c) What happens if you change the prior? Investigate how the posterior changes if $p(+)$ becomes much larger than $p(-)$ and vice versa. Make a sketch similar to (c).

(d) Let us assume that we decide for leftward motion whenever $r > \frac{1}{2}(\mu_+ + \mu_-)$. Interpret this decision rule in the above plots. How well does this rule do depending on the prior? What do you lose when you move from the full posterior to a simple decision (=decoding) rule?

(3) **Linear discriminant analysis (advanved):** Let us redo the calculations for the case of N neurons. If we write $\mathbf{r}$ for the vector of firing rates observed, then Bayes theorem now reads:

$$p(s|\mathbf{r}) = \frac{p(\mathbf{r}|s)p(s)}{p(\mathbf{r})} \quad .$$

We assume that the distribution of firing rates again follows a Gaussian so that

$$p(\mathbf{r}|s) = \frac{1}{(2\pi)^{N/2}\sqrt{\det C}} \exp\Big(- \frac{1}{2}(\mathbf{r} - \mathbf{m}_s)^T C^{-1} (\mathbf{r} - \mathbf{m}_s)\Big)$$

where $\mathbf{m}_s$ denotes the mean of the density for stimulus s, and C is the covariance matrix.

(a) Compute the log-likelihood ratio

$$l(\mathbf{r}) = \log \frac{p(+|\mathbf{r})}{p(-|\mathbf{r})} \quad .$$

(b) Assume that $l(\mathbf{r}) = 0$ is the decision boundary, so that any firing rate vector $\mathbf{r}$ giving a log-likelihood ratio larger than zero is classified as coming from the stimulus +. Compute a formula for the decision boundary. What shape does this boundary have?

(c) Assume that $p(+) = p(-) = 1/2$. Assume we are looking only at two neurons that are uncorrelated so that the covariance matrix is

$$C = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}$$

Sketch the decision boundary for this case.